The Prime Number Theorem

1.Introduction

The $+ve$ integers other than 1 may be devided into two classes.Prime numbers, which do not admit of resolution into smaller factors and composite numbers, which do.The prime numbers derive their peculiar importance from the 'fundamental theorem of arithmetic'.Euclid around 200 B.C. announced his theorem which states that there are infinitely many primes.This theorem naturally suggests the question of determining the no.of primes not exceeding a given $+ve$ real number x. To study this question we write $\pi(x)$ to denote the no.of primes $\leq x$,for a real no.$x \geq 1$.Needless to say, an exact answer in the form of an expression involving x is not available.After that it is natural to ask for a formula that approximates $\pi(x)$.The prime number theorem answers our question which says that $\pi(x) \sim \dfrac{x}{\log x}$.In other words $\lim_{x\to\infty} \dfrac{\pi(x)\log x}{x} = 1$.This statement is called 'the prime number theorem without error term'.

The prime number theorem was apparently first conjectured in the late 18th century,by Legendre and Gauss (independently).In particular Gauss conjectured an equivalent but more appealing form of the prime number theorem in 1792 which states that $\pi(x) \sim li(x)$ where $li(x) = \displaystyle\int_2^x \frac{dt}{\log t}$, which we call it as logarithmic integral.The prime number theorem proved much later by Hadamard and de la Vallee poussin in 1896,almost simultaneously but independently,following ideas introducing by Riemann in 1859.In summary the principle is to study the function $\zeta(s)$,defined for complex numbers s with $Re(s) > 1$,by the relation

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}. \tag{1}$$

Much before Riemann ,Euler had considered this function for real values of s.In effect,Euler recast the fundamental theorem of arithmetic which states that every natural number is expressible in unique way as a product of prime numbers,in terms of the following remarkable identity,called the *Euler product formula*.

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) = \prod_p \frac{1}{1 - \frac{1}{p^s}}. \tag{2}$$

for all real $s > 1$.Formally,the identity (2) is easy to verify by distributivity and the fundamental theorem of arithmetic and indeed (2) is equivalent to this theorem.We will recount this proof together with a justification for this identity for all complex numbers s with $Re(s) > 1$ later in this notes.

Euler as we have said ,studied the function $\zeta(s)$ for real values of s.Riemann,on the other

hand,showed that this function extends to a meromorphic function C whose only pole is a simple pole at $s = 1$ with residue 1 and this extended function satisfies a simple functional equation.He also observed that there is an explicit connection between the zeroes of $\zeta(s)$ extended as a meromorphic function on C and the distribution of prime numbers on account of the identity (2) viewed as a relation valid for all complex numbers s with $Re(s) > 1$.
Pursuing the direction shown by Riemann,Hadamard and de la Vallee poussin succeeded in proving the prime number theorem by first proving that there is no zero of $\zeta(s)$ on the line $Re(s) = 1$,which they then combined ,in quite different ways,with growth properties of $\zeta(s)$.
Following the work of Hadamard and de la Vallee Poussin and through the efforts of number of mathematicians such as E.Landau,G.H.Hardey,J.E.Littlewood much simpler proof of the prime number theorem were discovered,although these proofs still relied on growth properties of $\zeta(s)$ together with the meromorphic continuation of $\zeta(s)$ to all points of the $Re(s) \geq 1$ or a slightly larger region and the non-vanishing of $\zeta(s)$ in this region.
It thus came as surprise when,in 1931,S.Ikehara using ideas of N.Weiner,succeeded in deducing the prime number theorem without reference to the growth properties of $\zeta(s)$ and using only that $\zeta(s)$ admits a meromorphic continuation to all points in the region $Re(s) \geq 1$ and that $\zeta(s)$ does not vanish in this region.Finally,in 1933 Bochner,Landau and Heilbornn produced,using Ikehara's work,what is arguably the easiest route to the prime number theorem which we shall use in notes.
If we define the function $li(x)$ for $x \geq 2$ to be the integral $\int\limits_2^\infty \frac{dx}{\log x}$,we then have

$$\pi(x) = li(x) + \mathcal{O}\left(x \exp\left(-c(\log x)^{\frac{1}{2}}\right)\right). \tag{3}$$

This relation means that there are positive real numbers A and c such that

$$|\pi(x) - li(x)| \leq Ax \exp\left(-c(\log x)^{\frac{1}{2}}\right), \tag{4}$$

for all $x \geq 2$.The above equation is said to be 'the prime number theorem with error term'.
In these notes we take up the proof of Weiner-Ikehara in section(5),the sections preceding which describe the preliminaries required.

2.Equivalent forms of the Prime number theorem

In this section we consider a no.of asympotic relations that are equivalent to the prime number theorem $\pi(x) \sim \frac{x}{\log x}$.
We define the arithmetical function $\Lambda(n)$ called von Mangold function ,

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{for some prime } p \text{ and } m \in Z^+; \\ 0 & \text{otherwise} \end{cases}$$

Theorem: The following statements are equivalent.
(1).$\pi(x) \sim \frac{x}{\log x}$
(2).$\sum\limits_{p \leq x} \log p \sim x$, where the summation on the left hand side is running over all those *primes*

not exceeding x.

(3). $\sum_{n\leq x}\Lambda(n)\sim x$. where the summation on the left hand side is running over all those $+integers$ not exceeding x.

Proof: For our convenience we denote the sum in the (2) by $\theta(x)$ and the sum in (3) by $\psi(x)$.

(1) $\Rightarrow$ (2):Assume $\pi(x)\sim\dfrac{x}{\log x}$.

Since dropping a finite number of primes doesn't effect on the asympotic behaviour of $\pi(x)$ we can assume that prime numbers start from 3. Now,

$$\sum_{3\leq p\leq x}\log p\leq\left(\sum_{3\leq p\leq x}\mathbf{1}\right)\log x$$
$$\Rightarrow\theta(x)\leq\pi(x)\log x.$$

Taking limits as $x\to\infty$,

$$\limsup_{x\to\infty}\frac{\theta(x)}{x}\leq\limsup_{x\to\infty}\frac{\pi(x)\log x}{x}. \tag{5}$$

For $3\leq y\leq x$,

$$\log y\sum_{3\leq p\leq x}1\leq\log y\sum_{3\leq P\leq y}1+\sum_{y\leq p\leq x}\log p\leq\log y\pi(x)+\theta(x)$$

$$\log y\pi(x)\leq\log y\pi(y)+\theta(x).$$

Let $y=x^{\theta},\theta\in(0,1)$,

$$\theta\liminf_{x\to\infty}\frac{\pi(x)\log x}{x}\leq\theta\liminf_{x\to\infty}\frac{\log x\pi(x^{\theta})}{x}+\liminf_{x\to\infty}\frac{\theta(x)}{x}.$$

Since $\liminf_{x\to\infty}\dfrac{\log x\pi(x^{\theta})}{x}=0$, we will get

$$\liminf_{x\to\infty}\frac{\pi(x)\log x}{x}\leq\liminf_{x\to\infty}\frac{\theta(x)}{x} \tag{6}$$

$\therefore$ by (5) & (6), we have $\theta(x)\sim x$.

Similarly we can prove (2) $\Rightarrow$ (1).

(2) $\Leftrightarrow$ (3) :

We have

$$\begin{aligned}\psi(x) =& \sum_{n\le x}\Lambda(n)\\ =& \sum_{p^m\le x}\log p\\ =& \sum_{p\le x}\log p\left[\frac{\log x}{\log p}\right]\\ =& \sum_{1\le p\le x^{\frac{1}{2}}}\log p\left[\frac{\log x}{\log p}\right]+\sum_{x^{\frac{1}{2}}\le p\le x}\log p\\ =&\theta(x)-\theta(x^{\frac{1}{2}})+O(\log x\pi(x^{\frac{1}{2}}))\\ =&\theta(x)+O(x^{\frac{1}{2}}\log x).\end{aligned}$$

And hence

$$\frac{\psi(x)}{x}=\frac{\theta(x)}{x}+O\left(\frac{\log x}{x^{\frac{1}{2}}}\right).$$

As $x\to\infty$, we will get the desired result.In the last section we will prove the prime number theorem of the form $\psi(x)\sim x$.

3.Dirichlet series

A Dirichlet series is a series of the form $\sum_{n\ge1}\frac{a_n}{n^s}$, where a_n's are Real or Complex numbers.The Dirichlet series associated to an arithmetical function f is the series $\sum_{n\ge1}\frac{f(n)}{n^s}$,which we denote by $\zeta(f,s)$.

A word about notation:We denote a complex number by the letter s and the real and imaginary parts of s by σ and τ respectively.We will sometimes also use t to denote the imaginary part of s.The arithmetical function f which takes 1 at every natural number, we denote it by 1. The Dirichlet series associated to the arithmetical function 1 is the series $\sum_{n\ge1}\frac{1}{n^s}$. In place of writing $\zeta(1,s)$ for this series we simply write $\zeta(s)$.

Note that If the series $\sum_{n\ge1}\frac{a_n}{n^s}$ is absolutely convergent for $s=s_0=\sigma_0+i\tau_0$, it is then absolutely convergent for all s with $\sigma\ge\sigma_0$, as is evident from the following inequalities

$$\sum_{n\ge1}\left|\frac{a_n}{n^s}\right|=\sum_{n\ge1}\frac{|a_n|}{n^\sigma}\le\sum_{n\ge1}\frac{|a_n|}{n^{\sigma_0}}=\sum_{n\ge1}\left|\frac{a_n}{n^s}\right|.$$

It follows that if we define σ_a to be the infimum of all σ such that $\sum_{n\ge1}\frac{a_n}{n^s}$ is absolutely convergent for $s=\sigma$, then the series is absolutely convergent for all complex numbers s in the half plane determined by $Re(s)>\sigma_a$. We call σ_a the *abscissa of absolute convergence* of the Dirichlet series $\sum_{n\ge1}\frac{a_n}{n^s}$ and the half plane determined by $Re(s)>\sigma_a$ the *$half\ plane\ of\ absolute$ convergence*.

We know that the Dirichlet series $\sum_{n\geq1}\frac{1}{n^s}$ is absolutely convergent for $Re(s)>1$ and hence represent a holomorphic function in the half plane for which $Re(s)>1$ by Weierstrass theorem.So We can differentiate term by term.Then we will get $\zeta'(s)=-\sum_{n\geq1}\frac{logn}{n^s}$. we can easily show that $\zeta(\Lambda,s)$ and $\zeta'(s)$ are absolutely convergent for $Re(s)>1$ and hence they also represent holomorphic functions for $Re(s)>1$.The holomorphic function $\zeta(s)$ which is defined for $Re(s)>1$ is called the Riemann-zeta function.The proof the prime number theorem completely depend on the non vanishing of $\zeta(s)$ on the line $Re(s)=1$. We have a relation among these three holomorphic functions.

proposition: $-\zeta'(s)=\zeta(s)\zeta(\Lambda,s)$ for $Re(s)>1$.

Proof: Consider

$$\begin{aligned}\zeta(s)\zeta(\Lambda,s)=&\sum_{n\geq1}\frac{1}{n^s}\sum_{m\geq1}\frac{\Lambda(m)}{m^s}\\ =&\sum_{n\geq1}\sum_{m\geq1}\frac{\Lambda(m)}{(nm)^s}\\ =&\sum_{k\geq1}\frac{\sum_{nm=k}\Lambda(m)}{k^s}\\ =&\sum_{k\geq1}\frac{\sum_{d/k}\Lambda(d)}{k^s}\\ =&\sum_{k\geq1}\frac{\log k}{k^s}\\ =&-\zeta'(s).\end{aligned}$$

The rearrangement of terms in the proof being justified by the absolute convergent of $\zeta(s)$ and $\zeta(\Lambda,s)$ for $Re(s)>1$.

This implies that $\frac{-\zeta'(s)}{\zeta(s)}$ is a holomorphic in the half plane $Re(s)>1$ and hence that $\zeta(s)\neq0$ for all s in this half plane.

We can see that $\sum_{n\geq2}\frac{\Lambda(n)}{(logn)n^s}$ is absolutely convergent in the half plane $Re(s)>1$ and hence is a holomorphic function of s in this half plane.Let $L(\phi(s))$ denote $\frac{\phi'(s)}{\phi(s)}$ for a holomorphic function $\phi(s)$.For all $Re(s)>1$ we then have

$$L\left(\exp\left(\sum_{n\geq2}\frac{\Lambda(n)}{(\log n)n^s}\right)\right)=-\zeta(\Lambda,s).$$

On the other hand ,we have $L(\zeta(s))=-\zeta(\Lambda,s)$.Since half plane is simply connected and connected,it follows that there is a complex number $c\neq0$ such that $\zeta(s)=c\exp\left(\sum_{n\geq2}\frac{\Lambda(n)}{(\log n)n^s}\right)$ for all s in $Re(s)>1$.On letting $s\to+\infty$ along the real line,it is easily seen that $c=1$ and hence that

$$\zeta(s)=\exp\left(\sum_{n\geq2}\frac{\Lambda(n)}{(\log n)n^s}\right) \tag{7}$$

for all s in $Re(s) > 1$.
From the definition of $\Lambda(n)$ we see that $\dfrac{\Lambda(n)}{\log n}$ is $\dfrac{1}{k}$ when $n = p^k$ for an integer $k \geq 1$ and a prime p and is 0 for all other n.Therefore we have

$$\sum_{n\geq 2} \frac{\Lambda(n)}{(\log n)n^s} = \sum_p \sum_{k\geq 1} \frac{1}{kp^{ks}} = -\sum_p \log\left(1 - \frac{1}{p^s}\right) \tag{8}$$

for all s in $Re(s) > 1$. On combining the above two equations we obtain

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

for all s in $Re(s) > 1$, which is the Euler product formula.
Now define for any $n \in Z^+$ and for any real no.$x \geq 1$,

$$f(n,x) = \begin{cases} 0 & \text{if } x < n; \\ 1 & \text{if } x \geq n \end{cases}$$

and consider

$$\begin{aligned}
\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} &= \sum_{n\geq 1} \int_n^\infty \frac{\Lambda(n)s}{x^{s+1}} dx \\
&= s \sum_{n\geq 1} \int_1^\infty \frac{f(n,x)\Lambda(n)}{x^{s+1}} dx \\
&= s \int_1^\infty \sum_{n\geq 1} \frac{f(n,x)\Lambda(n)}{x^{s+1}} dx \\
&= s \int_1^\infty \sum_{n\leq x} \frac{\Lambda(n)}{x^{s+1}} dx \\
&= s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx.
\end{aligned}$$

$$\therefore \zeta(\Lambda, s) = s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx$$

for $Re(s) > 1$.

4.$\zeta(s) \neq 0$ for $Re(s) = 1$

To prove prime number theorem we must prove that $\zeta(s) \neq 0$ for$Re(s) \geq 1$. We already proved in the last section that $\zeta(s)$ has no zeroes for $Re(s) > 0$.But it remains to show that $\zeta(1+it) \neq 0$. In this section we will prove that the Riemann zeta function $\zeta(s)$ can be extended as a meromorphic function to the half plane for which $Re(s) > 0$ and it has no zeroes on the line $Re(s) = 1$.

Lemma: Let $\lambda_1, \lambda_2, \cdots$ be a real sequence which increases and has the limit infinity, and let $C(X) = \sum_{\lambda_n \le X} c_n$ where $\{c_n\}$ may be real or complex and the notation indicates a summation over the finite set of positive integers n for which $\lambda_n \le x$. Then, if $X \ge \lambda_1$and $\phi(x)$ has continuous derivative,we have

$$\sum_{\lambda_n \le X} c_n \phi(\lambda_n) = -\int_{\lambda_1}^{X} C(x)\phi'(x)dx + C(X)\phi(X).$$

Proof: Consider

$$\begin{aligned} C(X)\phi(X) - \sum_{\lambda_n \le X} c_n \phi(\lambda_n) &= \sum_{\lambda_n \le (X)} c_n \left(\phi(X) - \phi(\lambda_n)\right) \\ &= \sum_{\lambda_n \le X} \int_{\lambda_n}^{X} c_n \phi'(x)dx \\ &= \int_{\lambda_1}^{X} \sum_{\lambda_n \le x} c_n \phi'(x)dx \\ &= \int_{\lambda_1}^{X} C(x)\phi'(x)dx \end{aligned}$$

Hence the result follows.

Theorem(1): The Riemann zeta function $\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s}$ defined for $Re(s) > 1$ admits analytic continuation over the half plane $Re(s) > 0$ having as its singularity in this half plane a simple pole with residue 1 at $s = 1$.

Proof: In the above theorem ,take $\lambda_n = n$,$c_n = 1$,$\phi(x) = x^{-s}$ then $\sum_{n \le X} \frac{1}{n^s} = -\int_{1}^{X} \frac{[x]}{x^{s+1}} dx + \frac{[X]}{X^s}$ if $X \ge 1$.

Writing $[x] = x - (x)$, so that $0 \le (x) < 1$.So we obtain

$$\sum_{n \le x} \frac{1}{n^s} = \frac{s}{s-1} - \frac{s}{(s-1)X^{s-1}} - s\int_{1}^{X} \frac{(x)}{X^{s+1}dx} + \frac{1}{X^{s-1}} - \frac{(X)}{X^s}.$$

Now letting $X \to \infty$, we get

$$\zeta(s) = \frac{s}{s-1} - s\int_{1}^{\infty} \frac{(x)}{x^{s-1}} dx,$$

if $Re(s) > 1$.

Since $\left|\frac{x}{x^{s+1}}\right| < \frac{1}{x^{\sigma+1}}$ the last integral is uniformly convergent for $\sigma > \delta$,where δ is any fixed positive number and there fore represent a holomorphic function of s in $Re(s) > 0$.

This proves the theorem and the equation provides the continuation of $\zeta(s)$ over the half plane $re(s) > 0$.

Theorem(2): $\zeta(s)$ has no zeroes on the line $Re(s) = 1$.

Proof: We prove this theorem based on the following inequality

$$3 + 4cos\theta + cos2\theta \geq 0.$$

The above inequality holds for all real θ since the left hand side is nothing but $2(1 + cos2\theta)^2$. And also we have the following for $Re(s) > 1$

$$\log \zeta(S) = -\sum_{p} \log(1 - p^{-s}) = \sum_{p,m} \frac{1}{mp^{ms}}$$

By the equation in section(3). We have for $Re(s) > 1$,

$$log|\zeta(\sigma + i\tau)| = Re\sum_{n=2}^{\infty} c_n n^{-\sigma} - i\tau = \sum_{n=2}^{\infty} c_n n^{-\sigma} cos(t \log n)$$

where c_n is $\dfrac{1}{m}$if n is the $m'th$ power of prime,and 0 otherwise.Hence

$$log|\zeta^3(\sigma)\zeta^4(\sigma + i\tau)\zeta(\sigma + 2i\tau)| = \sum_{n\geq 2} c_n n^{-\sigma} 3 + 4cos(\tau \log n) + cos(2\tau \log n) \geq 0$$

since $c_n \geq 0$. Thus

$$|(\sigma - 1)\zeta(\sigma)|^3 \left|\frac{\zeta(\sigma + i\tau)}{\sigma - 1}\right|^4 |\zeta(\sigma + 2i\tau)| \geq \frac{1}{\sigma - 1}$$

holds for $\sigma > 1$.This shows that the point $1 + i\tau(\tau \gtrless 0)$ cannot be zero of $\zeta(s)$.
For ,if it were then since $\zeta(s)$ is regular at the points $1 + i\tau, 1 + 2i\tau$ and has simple pole at the point 1,the left hand side would tend to a finite limit and the right hand side tend to infinite when $\sigma \to 1 + 0$.
Hence we are done.

5.Taubarian theorem and proof of the PNT

In this section we will prove the prime number theorem in the form of $\psi(t) \sim t$ using the fact that $\zeta(s)$ is distinct from 0 at all points on the line $Re(s) = 1$,which we proved in the last section.The passage from this fact to the prime number theorem is effected by means of the following theorem ,which is a basic form of the $Weiner - Ikehara\ Tauberian\ theorem.$
We define Fourier transform of a function in C_c^∞ by

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}dx.$$

By this def.we can observe that $\displaystyle\int f\hat{g} = \int \hat{f}g$.
Let $\mathcal{D}(\mathbb{R})$ to denote the set of all C_c^∞ functions on $\mathbb{R}$ and $\mathcal{E}(\mathbb{R})$ to denote the subset of $\mathcal{D}(\mathbb{R})$ for which $\hat{\phi}(t) \geq 0$.

We write ψ to denote a fixed element of $\mathbb{R}$ normalised so that its support is contained in $(-1,1)$ and $\int \hat{\psi}(t) = 1$. Let $\psi_\lambda(t)$ to denote $\psi\left(\frac{t}{\lambda}\right)$ for each $\lambda > 0$.Then $\hat{\psi_\lambda}(t) = \lambda\hat{\psi_\lambda}(t)$ and $\int \hat{\psi_\lambda}(t)dt = 1$ for all $\lambda > 0$.

For each $l > 0$ we define $\delta_\lambda(l)$ by the relation

$$\delta_\lambda(l) = \int_{-\infty}^{-l} \hat{\psi_\lambda}(t)dt + \int_{l}^{\infty} \hat{\psi_\lambda}(t)dt \tag{9}$$

Since $\hat{\psi}(t)$ is integrable on $\mathbb{R}$ we see that for a fixed $\lambda > 0$,$\delta_\lambda(l)$ tends to 0 as $l \to +\infty$ and for a fixed $l > 0$,$\delta_\lambda(l)$ tends to 0 as $\lambda \to +\infty$.

Finally write d^*t to denote $\frac{dt}{t}$,the Haar measure on $\mathbb{R}^*$ and write e_+^t to denote the function which is e^t when $t > 0$ and 0 when $t < 0$.

Theorem: Let f be a positive increasing function on $\mathbb{R}_+^*$ such that $\int_1^\infty f(t)d^*t$ has a finite abscissa of convergence a. Suppose that there is a real number c_a such that

$$\int_1^\infty f(t)t^{-s}d^*t - \frac{c_a}{s-a} \tag{10}$$

extends continuously to the closed strip $Re(s) \geq a$ and $|Im(s)| \leq T$ for some $T > 0$.

For all Λ in $(0,T)$, we then have that

$$\lim_{x\to\infty} \int_0^\infty f(e^t)e^{-at}\hat{\psi_\lambda}(t-x)dt = c_a. \tag{11}$$

Proof: Let us write $h(s)$ to denote $\int_1^\infty f(t)t^{-s}d^*t - \frac{c_a}{s-a}$ and its continuous extension.Thus $h(s)$ is analytic for $Re(s) > a$,and is continuous on $Re(s) \geq a$ and $|Im(s)| \leq T$.

Further, when $\sigma > a$ the functions $f(e^t)e_t^{-\sigma t}$ and $e_+^{-(\sigma-a)t}$ are integrable on $\mathbb{R}$ and we have

$$h(\sigma + i\tau) = \int_1^\infty (f(t)t^{-\sigma} - c_a t^{-(\sigma-a)})t^{-i\tau}d^*t = \mathcal{F}(f(e^t)e_+^{-\sigma t} - c_a e_+^{-(\sigma-a)t})(\tau) \tag{12}$$

for every $\sigma > a$ and all t in $\mathbb{R}$. Thus for each $\sigma > a$ and each ϕ in $\mathcal{D}(\mathbb{R})$ we deduce using that

$$\int_0^\infty (f(e^t)e^{-\sigma t} - c_a e^{-(\sigma-a)t})\hat{\phi}(t)dt = \int_{-\infty}^\infty h(\sigma+i\tau)\phi(\tau)d\tau. \tag{13}$$

Let x be any real number ≥ 1. and let $\lambda \in (0,T)$. When $\psi_\lambda(t)$ is in $\mathcal{D}(\mathbb{R})$ so is $\psi_\lambda(t)e^{ixt}$ and $\mathcal{F}(\psi_\lambda(t)e^{ixt})$ is $\hat{\psi_\lambda}(t-x)$. Applying (13) to $\phi(t)e^{ixt}$ in place of $\phi(t)$ and passing to the limit in (13) as σ tends a with $\sigma > a$ we obtain

$$\lim_{\sigma\to a} \int_0^\infty (f(e^t)e^{-(\sigma-a)t})\hat{\psi_\lambda}(t-x)dt = \lim_{\sigma\to a} \int_{-\infty}^\infty h(\sigma+i\tau)\psi_\lambda(\tau)e^{ix\tau}d\tau. \tag{14}$$

Since f and $\hat{\psi}_\lambda$ are positive functions and $\sigma > a$,the limit and integral signs on the left hand side of (14) may be interchanged by the monotone convergence theorem.Since $h(s)$ is continuous on $Re(s) \geq a$ and $|Im(s)| \leq T$ and the support of ψ_λ is in $(-T,T)$,$h(\sigma+i\tau)\phi(\tau)$ converges uniformly in τ to $h(a+i\tau)\phi(\tau)$ as σ tends to a with $\sigma > a$. Thus the limit and integral signs on the right hand side of (14) may interchanged.Consequently,

$$\int_0^\infty f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt = c_a \int_0^\infty \hat{\psi}_\lambda(t-x)dt + \int_{-\infty}^\infty h(a+i\tau)\psi_\lambda(\tau)e^{ix\tau}d\tau. \tag{15}$$

On making the change of variable $t-x \to t$ in the first integral on the right hand side of (15) and recalling that $\hat{\psi}_\lambda(t)dt = 1$,we obtain

$$\int_0^\infty f(e^t)e^{-at}\hat{\psi}_\lambda t - xdt = c_a + c_a \int_{-\infty}^{-x} \hat{\psi}_\lambda(t-x)dt + \int_{-\infty}^\infty h(a+i\tau)\psi_\lambda(\tau)e^{ix\tau}d\tau \tag{16}$$

for all $\lambda \in (0,T)$.The function $h(a+i\tau)\psi_\lambda(\tau)e^{ix\tau}$ is continuous and of compact support and thus integrable on $\mathbb{R}$.Thus the third term on the right hand side of (16) tends to 0 as x tends $+\infty$,by the Riemann -Lebesgue lemma.Since $\hat{\psi}_\lambda(t)$ is integrable on $\mathbb{R}$,the second term on the right hand side of (16) also tends to 0 as x tends to $+\infty$.Thus the theorem follows on passing to the limit in (16) as x tends to $+\infty$.

Proposition: Let f be a positive increasing function on $(1,\infty)$ and a a real number.Suppose that for some $\Lambda > 0$ and a real number c_a we have

$$\lim_{x\to+\infty} \int_0^\infty f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt = c_a. \tag{17}$$

Then the function $f(e^x)e^{-ax}$ is bounded above on $\mathbb{R}_+^*$. Moreover ,if K is an upper bound for $f(e^x)e^{-ax}$ on $\mathbb{R}_+^*$,then for all $l > 0$ we have

$$\frac{c_a - K\delta_\lambda(l)}{e^{2al}} \leq \liminf_{x\to\infty} f(e^x)e^{-ax} \leq \limsup_{x\to\infty} f(e^x)e^{-ax} \leq \frac{c_a e^{2al}}{1-\delta_\lambda(l)}. \tag{18}$$

Proof: Now, for any $l \leq x$We have

$$\begin{aligned}\int_0^\infty f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt \geq &\int_{x-l}^{x+l} f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt \\ \geq &f(e^{x-l})e^{-a(x+l)}\int_{x-l}^{x+l} \\ = &f(e^y)e^{-ay}e^{-2al}\int_{-l}^{l}\hat{\psi}_\lambda(t)dt\end{aligned}$$

$$\therefore c_a \geq \limsup_{y\to\infty} f(e^y)e^{-ay}e^{-2al}(1-\delta_\lambda(l))$$

$$\frac{c_a e^{2al}}{1-\delta_\lambda(l)} \geq \limsup_{x\to\infty} f(e^x)e^{-ax}.$$

And hence $f(e^x)e^{-ax}$ bounded above by K say . Now

$$\begin{aligned}\int_0^{x-l} f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt + \int_{x+l}^{\infty} f(e^t)e^{-at}\hat{\psi}_\lambda(t-x)dt \leq & K(\int_{-\infty}^{x-l} \hat{\psi}_\lambda(t-x)dt + \int_{x+l}^{\infty} \hat{\psi}_\lambda t - xdt) \\ =& K\int_{-\infty}^{-l} \hat{\psi}_\lambda(t)dt + \int_{l}^{\infty} \hat{\psi}_\lambda(t)dt \\ =& K\delta_\lambda(l)\end{aligned}$$

$$\therefore \liminf_{x\to\infty} \int_{x-l}^{x+l} f(e^t)e^{-at}\hat{\psi}_\lambda(t-x) \geq c_a - K\delta_\lambda(l)$$

$$\liminf_{x\to\infty} f(e^{x+l})e^{-a(x-l)} \int_{-l}^{l} \hat{\psi}_\lambda(t)dt \geq c_a - K\delta_\lambda(l)$$

$$\Rightarrow f(e^{x+l})e^{-a(x-l)} \geq c_a - K\delta_\lambda(l)$$

$$f(e^x)e^{-ax} \geq \frac{c_a - K\delta_\lambda(l)}{e^{2al}}$$

Hence the theorem follows.

Proof of prime number theorem: Let us take ψ in place of f in the above argument.Since $\int_1^\infty \psi(x)x^{-(s+1)}dt = \frac{\zeta(\lambda,s)}{s} = \frac{-\zeta'(s)}{s\zeta(s)}$ has abscissa of convergence 1 and $\frac{-\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1} = \frac{-1}{s}\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) - \frac{1}{s}$ extends continuously to the closed strip $Re(s) \geq 1$ and $Im(s) \leq T$ for every $T > 0$ (since $\zeta(s) \neq 0$ on the line $Re(s) = 1$.)
Then by the above theorem,for any $\lambda \in (0,T)$ we have

$$\frac{1-K\delta_\lambda(l)}{e^{2l}} \leq \liminf_{x\to\infty} \psi(e^x)e^{-x} \leq \limsup_{x\to\infty} \psi(x)e^{-x} \leq \frac{e^{2l}}{1-\delta_\lambda(l)}$$

The above equation holds for any l, for every $\lambda \in (0,T)$ and T can be arbitarly large so that we can tend λ to ∞ . Then we have

$$1 \leq \liminf_{x\to\infty} \psi(e^x)e^{-x} \leq \limsup_{x\to\infty} \psi(e^x)e^{-x} \leq 1 \tag{19}$$

By making change of variable the prime number theorem follows.