The Riemann mapping theorem

1 Introduction

In this short note, we give a complete and self-contained proof of the most profound and important theorem in Complex Analysis:

Theorem 1. *Let $U \subset \mathbb{C}$ be a simply-connected domain, $U \neq \mathbb{C}$. Then we can find a biholomorphism from U onto the unit disk $\mathbb{D}$.*

Remark 2. Throughout these notes, a *domain* is an open and connected subset of $\mathbb{C}$. Unless otherwise specified, U will denote a domain in $\mathbb{C}$.

To say that the above theorem is shocking is an understatement! To get a sense of the profundity of the above theorem, let us first clarify the meanings of the terms *simply-connected* and *biholomorphism.*

Definition 3. Let U be a domain. Let $\gamma_0, \gamma_1 : [0, 1] \to U$ be two continuous curves that have the same starting and ending points, say a and b, respectively. We say that γ_0 and γ_1 are *homotopic in* U if γ_0 can be continuously deformed into γ_1. More precisely, there exists a continuous map $H : [0, 1]\times[0, 1] \to U$ such that

$$
\begin{aligned}
H(s, 0) &= \gamma_0(s) && \forall s \in [0, 1] \\
H(s, 1) &= \gamma_1(s) && \forall s \in [0, 1] \\
H(0, t) &= a && \forall t \in [0, 1] \\
H(1, t) &= b && \forall t \in [0, 1].
\end{aligned}
$$

We say that U is *simply-connected* if any two curves in U that have the same starting and ending points are homotopic.

Remark 4. We have considered only curves defined on $[0, 1]$. However, it is quite easy to reformulate the definition for curves defined on any interval $[a, b]$.

Exercise 5. Show that being homotopic in U is an equivalence relation.

Remark 6. The use of the variable 't' for the second-slot of H is not coincidental. It is helpful to view this variable as time and adopt a "dynamical" viewpoint. We view the curve γ_0 as getting slowly deformed over time into γ_1. To emphasize this view, we will often think of the homotopy H as a continuous one parameter family of curves γ_t.

Example 7. Let U be a convex domain. Then U is simply-connected. To see this consider the straight line homotopy between the curves γ_0 and γ_1 given by

$$\gamma_t(s) := (1 - t)\gamma_0(s) + t\gamma_1(s).$$

Example 8. The set $\mathbb{C} \setminus \{0\}$ is not simply-connected. This is intuitively clear. You will have all the tools needed to give a rigorous proof

Exercise 9. Show that a domain U is simply-connected if and only if every loop in U is homotopic to a constant curve.

Definition 10. An injective holomorphic map from a domain is called a *biholomorphism.*

Exercise 11. If $f : U \to \mathbb{C}$ is a biholomorphism, show that f' is nowhere vanishing on U.

Exercise 12. Let U be a domain and let $f : U \to \mathbb{C}$ be a biholomorphism. Show that $f^{-1} : f(U) \to U$ is also holomorphic.

The last exercise shows that any biholomorphism is automatically a homeomorphism onto its image. Biholomorphisms are also known as *conformal isomorphisms.*

Exercise 13. Let U be a convex domain. Show that U is homeomorphic to the unit disk $\mathbb{D}$.

The Riemann mapping theorem asserts something far stronger than the above exercise. It says, in particular, that the homeomorphism that exists from the exercise can be chosen to be a biholomorphism. The interior of the square is a certainly a convex domain. However, writing down an explicit biholomorphism from the interior of the square into the disk is very difficult (try it!) and cannot even be done in terms of elementary functions. The profundity of the Riemann mapping should be apparent now. An arbitrary simply connected domain could be highly complicated and it is not at all obvious that it is even *homeomorphic* to the unit disk! Once we have a biholomorphism to the unit disk, we can carry out our analysis in $\mathbb{D}$ and transfer the result back to the original domain via the biholomorphism.

The above paragraph motivates the question as to why one should even believe that the Riemann mapping theorem is true?! Riemann's intuition came from physics; specifically hydrodynamics and electrodynamics. We will not delve into this. See [BN10] for a detailed and elementary treatment. Though Riemann's intuition was correct, his proof was not. Riemann used physics to justify a theorem, now known as the Dirichlet principle which was the crucial tool used in his proof. Weierstrass gave a counterexample to the Dirichlet principle proving that Riemann's proof was incorrect.

Riemann's original approach can be fixed. See [GK17]. However, we will **not** be following this approach. We will instead present the elegant approach due to Koebe.

2 Prerequisites

Here we will state (without proof) standard facts from complex analysis that we will assume the reader is familiar with. The textbooks [BN10, Gam01, SS03] are excellent elementary references for these results. For a more terse and advanced treatment, see [NN01]

Theorem 14 (Fundamental Theorem of Complex Calculus)**.** *Let f be holomorphic on U and suppose F is a primitive (anti-derivative) of f, i.e., $F' = f$. Then for any piecewise smooth curve $\gamma : [a, b] \to U$, we have*

$$\int_\gamma f = F(b) - F(a).$$

Theorem 15 (Cauchy's theorem for the disk)**.** *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and let $\gamma : [a, b] \to \mathbb{D}$ be a piecewise-smooth closed loop. Then*

1. *f has a primitive, i.e., we can find $F : \mathbb{D} \to \mathbb{C}$ such that $F' \equiv f$.*
2. *$\int_\gamma f dz = 0$.*

Theorem 16 (The Argument Principle)**.** *Let f be meromorphic on some neighborhood of the closure of a disk D. Assume that f has no zeros or poles on ∂D. Then*

$$\int_{\partial D} \frac{f'(z)}{f(z)} dz = N - P,$$

where N is the number of zeros of f in D and P is the number of poles in D.

Theorem 17 (Open mapping theorem)**.** *Any non-constant holomorphic mapping on a domain U is an open map.*

3 The Schwarz lemma

The Schwarz lemma is one of the most important "lemmas" in all of mathematics. There is an entire book by Sean Dineen on the lemma! Therefore we have no qualms in calling it a theorem.

Theorem 18. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and suppose $f(0) = 0$. Then*

1. $|f(z)| \leq z$,
2. $|f'(0)| \leq 1$,
3. *If either of the inequalities above is not strict, then f is a rotation.*

Proof. Consider the function $g(z) = f(z)/z$ with the value at 0 defined to be $f'(0)$. The function g is clearly holomorphic except for 0 and as g is clearly continuous at 0 also, 0 is a removable singularity for g. If $|z| = r$, then $|g(z)| < 1/r$ which by the maximum-modulus theorem propagates to the whole disk of radius r. By taking $r \to 1^+$, we conclude that $|g(z)| \leq 1$ with equality if and only if g is constant. Hence, $|f(z)| \leq |z|$ with equality if and only $f = cz$ where c is a unimodular constant, in other words when f is a rotation. The other assertions follows easily. □

Exercise 19. Fix $a \in \mathbb{D}$. Show that the fractional linear transformation

$$\psi_a := \frac{a - z}{1 - \overline{a}z}$$

that interchanges 0 and a is in fact an automorphism of the disk.

Theorem 20. *Let ψ be an automorphism of the unit disk and let $a = \psi(0)$. Then we can find θ such that*

$$\psi = e^{i\theta} \frac{e^{i\theta} a - z}{1 - \overline{e^{i\theta} a} z}.$$

Proof. Consider the map $\phi := \psi_a \circ \psi$. Now $\phi(0) = 0$ and ϕ is an automorphism and therefore ϕ is a rotation. This proves the theorem. □

4 A Cauchy theorem for simply-connected domains

Theorem 21. *Let U be a domain and $f : U \to \mathbb{C}$ be holomorphic. Suppose $\gamma_1, \gamma_2 : [0, 1] \to U$ are homotopic piecewise smooth curves. Then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz,$$

whenever $\gamma_i : [a, b] \to U$ is a piecewise smooth curve.

Proof. Let $H : [0,1] \times [0,1] \to U$ be a homotopy from γ_0 to γ_1. Though the curves γ_0 and γ_1, there is absolutely no reason to expect that all the intermediate curves are also piecewise smooth. However, we shall prove the result with this added assumption. The set $K := H([0,1] \times [0,1])$ is a compact set. Let $\varepsilon > 0$ be chosen so small that the disk of radius 3ε centered at any point in K is fully-contained in U.

Exercise 22. Show that $\varepsilon = d(K, \partial U)/6$ works.

Any continuous function on a compact set is uniformly continuous. Consequently, we can find a $\delta > 0$ such that if $|t_1 - t_2| < \delta, t_1, t_2 \in [0,1]$ then

$$|\gamma_{t_1}(s) - \gamma_{t_2}(s)| < \varepsilon.$$

Fix $t_1 \in [0,1]$ such that $|t_1| < \delta$. Cover $\mathrm{im}(\gamma_0)$ by finitely many disks $\{D_0, D_1, \dots, D_n\}$ centered at points of $\mathrm{im}(\gamma_0)$ of radius 2ε such that disks of radius ε centered at the same points also cover $\mathrm{im}(\gamma_0)$. This can be done because of compactness. By re-indexing these disks, we can assume that $\gamma_0(0) = \gamma_{t_1}(0) \in D_0$ and $\gamma_0(1) = \gamma_{t_1}(1) \in D_n$. Choose points $z_0, z_1, \dots, z_{n+1}$ and $w_0, w_1, \dots, w_{n+1}$ such that

- $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$,
- $z_i \in \mathrm{im}(\gamma_0)$ and $w_i \in \mathrm{im}(\gamma_{t_1})$,
- $z_0 = w_0 = \gamma_0(0) = \gamma_{t_1}(0)$,
- $z_{n+1} = w_{n+1} = \gamma_0(1) = \gamma_{t_1}(1)$,
- By re-indexing, we may assume that $z_0, \dots, z_{n+1}$ and $w_0, \dots, w_{n+1}$ are consecutive points on γ_{t_1} and γ_{t_2}.

On each D_i, let F_i be a primitive of f. On $D_i \cap D_{i+1}$, the primitives F_i and F_{i+1} must differ by a constant. Consequently,

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}).$$

Rearranging, we get

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1})$$

The above equation combined the fundamental theorem of calculus for complex line integrals shows that

$$\begin{aligned}
\int_{\gamma_1} f - \int_{\gamma_2} f &= \sum_{i=0}^{n} F_i(z_{i+1}) - F_i(z_i) - \sum_{i=0}^{n} F_i(w_{i+1}) - F_i(w_i) \\
&= \sum_{i=0}^{n} F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i)) \\
&= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) = 0.
\end{aligned}$$

What we shown is that if the curve γ_{t_1} is suitably close to the curve γ_0, then the integrals of f over γ_0 and γ_{t_1} are the same. Now, we can repeat the same argument with γ_{t_1} playing the role of γ_0 and γ_{t_2} that of γ_{t_1}, where $|t_2 - t_1| < \delta$. Repeating this finitely many times yields the result. □

Exercise 23. Think about how the above argument can be modified to prove the above theorem without the added assumption that γ_t is a always piecewise smooth.

Corollary 24 (Cauchy's theorem)**.** *Let f be holomorphic on U and let γ be a piecewise smooth curve that is homotopic to a constant curve. Then*

$$\int_\gamma f = 0.$$

In particular, if U is simply-connected, then

$$\int_\gamma f = 0,$$

whenever γ is a piecewise smooth loop in U.

Exercise 25. Show that on any simply connected domain, any holomorphic function has a primitive.

Exercise 26. Show that $\mathbb{C} \setminus \{0\}$ is not simply-connected.

4.1 The complex logarithm

The final exercise in the previous section allows to answer the following natural question: On which domains can we find a holomorphic branch of the complex logarithm?

Such a domain must necessarily miss the origin. We now give a sufficient condition.

Theorem 27. *Let U be a simply-connected domain that misses the origin and $1 \in U$. Then we can find a holomorphic function F on U with the following properties:*

1. $\exp(F(z)) = z$.
2. $F(r) = \log r$ *whenever r is a real number near* 1.

Proof. If a branch of the logarithm where to exist then its derivative must be $\frac{1}{z}$. So we define

$$F(z) := \int_\gamma \frac{1}{z},$$

where γ is any curve from 1 to z. The fact that this is well-defined, holomorphic and in fact a primitve of $\frac{1}{z}$ follows exactly as in the proof of Exercise 25. Consider $G(z) := z\exp(-F(z))$, then

$$\frac{dG}{dz} = \exp(-F(z)) - zF'(z)\exp(-F(z)) = 0.$$

It follows that G is a constant. Clearly $F(1) = 0$ from which it follows that $G(z) \equiv 1$ proving that $\exp(F(z)) = z$. If r is a real number close to 1 and $r \in U$, then

$$F(z) = \int_1^r \frac{dx}{x} = \log r.$$

□

5 Montel's thoerem

We will now study properties of families of holomorphic functions.

Definition 28. Let $\mathcal{F}$ be a family of continuous functions on U. We say that $\mathcal{F}$ is

1. *uniformly bounded on compacts* if for each compact set $K \subset U$ there is a constant C_K such that

$$|f(z)| < C_k, \quad z \in K, f \in \mathcal{F}.$$

2. *equicontinuous* if for each compact set $K \in U$ and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(z_1) - f(z_2)| < \varepsilon,$$

   whenever $z_1, z_2 \in K, |z_1 - z_2| < \delta$;
3. *normal* if for each sequence f_n in $\mathcal{F}$ there is a holomorphic function $f : U \to \mathbb{C}$ such that on each compact set $K \subset U$, $f_n \to f$ uniformly.

Remark 29. It is easy to formulate the above definitions for arbitrary topological spaces.

Example 30. The standard example of an equicontinuous family is the set of holomorphic functions on the unit disk whose derivatives are uniformly bounded on $\mathbb{D}$. On the other hand, the family $\{z^n\}$ defined on any open set containing the unit circle is not an equicontinuous family. Neither is it a normal family. But it is uniformly bounded.

The above definitions might seem *ad hoc* and without adequate motivation. For a detailed treatment, please refer to the textbook by Conway [Con73]. The theorem we would require is from the Ph.D thesis of Montel. This remarkable theorem kick-started the subject now known as complex dynamics which is one of the "hot" research areas.

Theorem 31. *Let $\mathcal{F}$ be a uniformly bounded family of holomorphic functions defined on U. Then*

1. *$\mathcal{F}$ is an equicontinuous family.*
2. *$\mathcal{F}$ is a normal family.*

Remark 32. The first conclusion of the theorem is not true for real-analytic functions. Consider the family of functions $\{\sin(nx)\}$. However, the famous Ascoli-Arzela theorem asserts that a uniformly bounded and equicontinuous family is always normal. Holomorphicity is thus not needed in the proof of the second conclusion.

Proof. Let $K \subset U$ be an arbitrary compact set and let $r = \frac{d(K, \partial U)}{6}$. Let $z, w \in K$ with $|z - w| < r$ and let γ be the standard parametrization of $\partial D_{2r}(w)$. Note that both z and w are in $D_{2r}(w)$ and in turn the closure of this disk is a subset of U. Thus by Cauchy's integral formula

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right] d\zeta.$$

Note that

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \frac{|z - w|}{|\zeta - z||\zeta - w|} \leq \frac{|z - w|}{r^2},$$

where the inequality follows because both $|\zeta - w|$ and $|\zeta - z|$ are lesser than r. Let M be the unfirom bound for the family $\mathcal{F}$ corresponding to the set of points in U which is at a distance $\leq 2r$ from K. Then by the ML-inequality, we get

$$|f(z) - f(w)| \leq \frac{1}{2\pi}\frac{2\pi r}{r^2}M|z-w|.$$

In short,

$$|f(z) - f(w)| < C|z-w| \;\; \forall f \in \{,\; z, w \in K, |z-w| < r.$$

This shows that $\mathcal{F}$ is an equicontinuous family.

For the proof of the second part, let $\{K_l\}_{\ell=0}^{\infty}$ be an exhaustion by compacts of U, i.e.,

- $K_\ell \subset \mathrm{int}(K_{\ell+1})$,
- $U = \bigcup_{\ell=1}^{\infty} K_\ell$.

Exercise 33. Show that any domain in $\mathbb{R}^n$ admits an exhaustion by compacts.

Let $\{w_j\}_{j=1}^{\infty}$ be a sequence of points that is dense in U. We can extract subsequences

$$\begin{array}{l} f_{1,1}, f_{1,2}, f_{1,3} \cdots\cdots \\ f_{2,1}, f_{2,2}, f_{2,3} \cdots\cdots \\ f_{3,1}, f_{3,2}, f_{3,3} \cdots\cdots \\ f_{4,1}, f_{4,2}, f_{4,3} \cdots\cdots \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \end{array}$$

with the property that

- Each row is a subsequence of the preceding row,
- the sequence $\{f_{n,k}(w_n)\}_{k=1}^{\infty}$ converges.

The diagonal subsequence $g_n := f_{n,n}$ clearly converges on the entire set $\{w_n\}$. Let $K \subset U$ be a compact set. We will show uniform convergence of g_n on K. Fix $\varepsilon > 0$ and let $\delta > 0$ be as per the definition of equicontinuity for the family $\mathcal{F}$. We can find points $w_1, \ldots, w_k$ such that the disks $D_\delta(w_k)$ cover K. If $z \in K$, then $z \in D_\delta(w_j)$ for some j. Hence

$$|g_n(z) - g_m(z)| \leq |g_n(w_j) - g_n(z)| + |g_m(w_j) - g_n(w_j)| + |g_m(z) - g_m(w_j)| < 3\varepsilon,$$

when n, m are suitably large. This shows that g_n's are uniformly Cauchy and hence converge uniformly to a map $g : U \to \mathbb{C}$.

Exercise 34. Use Morera's theorem to show that the uniform limit of a sequence of holomorphic mappings is holomorphic.

Observe that our choice of g_n now depends on the compact set. Now the exhaustion by compacts come into the picture.

Exercise 35. Complete the proof by applying another diagonal argument.

□

Theorem 36. *Let $f_n : U \to \mathbb{C}$ be a sequence of injective holomorphic functions that converge uniformly on compacts to a holomorphic map f. Then either f is constant or injective.*

Proof. Suppose f is not injective. Let $z_1, z_2 \in U$ be distinct points such that $f(z_1) = f(z_2)$. The new sequence $g_n(z) := f_n(z) - f_n(z_1)$ is a sequence of holomorphic maps that converge uniformly on compacts to the holomorphic map $g(z) := f(z) - f(z_1)$. If $g \equiv 0$ then we are done. Otherwise, z_2 is an isolated zero of g. Observe that by our hypothesis of injectivity of f_n, z_1 is the unique zero of each f_n. By the argument principle

$$1 = \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz,$$

where γ is a small circle centered at z_2 chosen in such a manner that g has no zero on $\mathrm{im}(\gamma)$ and no zero besides z_2 in the interior of γ. The sequence $1/g_n$ converges uniformly to $1/g$ on γ and $g_n' \to g'$ uniformly on γ. Consequently

$$0 = \frac{1}{2\pi i} \int_\gamma \frac{g_n'(z)}{g_n(z)} dz \to \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz = 1,$$

a contradiction. □

6 Proof of the Riemann mapping theorem

Step 1. *Mapping into a bounded subdomain of* $\mathbb{D}$. By hypothesis, we can find a point $a \in \mathbb{C} \cap (\mathbb{C} \setminus D)$. The map $z \mapsto (z - a)$ maps D biholomorphically onto a domain that misses 0. Consider the map

$$f(z) := \log(z - a).$$

This means that $\exp(f(z)) = z - a$. Consequently, if z and w are distinct points in D, it follows that $f(z) \neq f(w) + 2n\pi i$ for any $n \in \mathbb{Z}$. Fix $w \in D$. Suppose there is a sequence $z_n \to w$ such that $f(z_n) \to f(w) + 2\pi i$. Exponentiating, we see that $z_n \to w$ and therefore $f(z_n) \to f(w)$, a contradiction. This shows that we can find an open set N around $f(w) + 2\pi i$ such that $f(D) \cap N = \emptyset$. Define

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}.$$

Now f is injective, and F is just the composition of f with a fractional linear transformation and consequently $F : D \to F(D)$ is a biholomorphism. As f avoids N, it follows that $F(D)$ is a bounded set which we may assume, by rescaling, is an open subset of $\mathbb{D}$ that contains 0.

Step 2. *Constructing the required biholomorphism.* We will assume that D is a simply-connected open subset of $\mathbb{D}$. Let

$$\mathcal{F} := \{f : D \to \mathbb{D}, f \text{ is holomorphic and injective with } f(0) = 0\}.$$

By Step 1., $\mathcal{F}$ is a non-empty set and by Montel's theorem it is a normal family. Let

$$m = \sup\{|f'(0)| : f \in \mathcal{F}\}.$$

Observe that the identity map certainly is in $\mathcal{F}$ and hence $m \geq 1$. Let $f_n \in \mathcal{F}$ be a sequence such that $|f_n'(0)| \to m$. By normality, we may assume, by passing to a subsequence, that $f_n \to f : D \to \overline{D}$. Now, $f(0) = 0$ and also $f'(0) \neq 0$ (why?!). This means that f is non-constant and therefore by the open mapping theorem, $f(D) \subset \mathbb{D}$. Moreover, by Hurwitz's theorem, it follows that f is injective.

Step 3. *Finishing the proof.* We have to show that f is surjective. Suppose f misses a point b in $\mathbb{D}$. Let ψ_α be the automorphism of $\mathbb{D}$ that interchanges 0 and α where $\alpha \in \mathbb{D}$. Let $\Omega := (\psi_b \circ f)(D)$. Note that

Ω is simply-connected and misses 0. We can find a holomorphic branch of the square-root function on Ω. Consider the map

$$F := \psi_{g(b)} \circ g \circ \psi_b \circ f.$$

The map F is clearly injective and $F(0) = 0$. Let h denote the squaring function, we have

$$\psi_b \circ h \circ \psi_{g(b)} \circ F = f.$$

Now, $\Phi := \psi_b \circ h \circ \psi_{g(b)}$ is clearly a self-map of $\mathbb{D}$ that fixes 0 and is *not* an automorphism of a disk. Consequently, $|\Phi'(0)| < 1$ which means that $|F'(0)| > |f'(0)|$, a contradiction and we are done.

Remark 37. The astute reader would have observed that we have actually used only the fact that the domain D admits a branch of the logarithm. We have actually shown that the topological notion of simply-connected is actually equivalent to the analytic requirement of admitting an analytic branch of the logarithm!

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